The Calculation of Occluded Radiative Exchange Configuration Factors

by
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Abstract

The calculation of illuminance at a point in many lighting system simulations assumes that architectural surfaces are homogeneous diffuse emitters. If the point has an unoccluded view of an illuminating surface, the calculation is straightforward. It becomes considerably more complicated if the illuminating surface is partially occluded. We describe a new procedure for determining illuminance in the presence of occluding surfaces. The flux from the illuminating surface is considered to consist of differentially divergent beams, some of which project onto the plane containing the illuminated point and produce patches of uniform differential illuminance. If the illuminated point is within a patch, the beam contributes to its illuminance. A relatively simple step function is found that weights the differential illuminance with 1 or 0, depending on whether the point is inside or outside a beam's projection. This permits integration over all the beams, resulting in the final illuminance at the point. Occluding surfaces are accounted for by permitting them to block beams and thus alter a beam's projection. Integration over all beam directions gives the occluded illuminance. The final result can be interpreted in terms of rays cast from the illuminated point. It is found that a system of surfaces establishes a unique set of rays and only those that strike the source need be considered. Each such intersection produces an increment to the illuminance, calculated from a function that results from the integration over all beams. The result is an efficient method to account for occluded illuminance that shows an unexpected connection between radiative transfer analysis and ray casting.

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Introduction and Strategy

The calculation of illuminance from homogeneous diffuse area sources with arbitrary polygonal shapes was first performed by Lambert.[1] His equation, rediscovered many times in many forms, assumes that the illuminated point has an unoccluded view of the illuminating polygon. In the presence of occluding objects, the computation becomes significantly more complicated. Various simple procedures have been used to account for occlusion, based either on surface intersection of cast rays,[2] or projection of occluding surfaces onto the emitter.[3] Recently, Paulin,[4] Arvo,[5] DiLaura,[6] Gould,[7] and Seider[8] have worked on more general approaches to this problem.

Beginning with previous developments, [4-6] we use the following strategy. Flux from a polygonal source is considered to consist of an infinity of differentially divergent beams, some of which project onto the plane containing the illuminated point, producing projected areas of uniform differential illuminance. If the illuminated point is within a projection, the beam contributes to its illuminance. A relatively simple step function is found that weights the differential illuminance with 1 or 0, depending on whether the point is inside or outside a beam's projection. This permits integration over all the beams, resulting in the final illuminance at the point. Occluding surfaces are accounted for by permitting them to block beams and alter the shape of the differentially illuminated projections. Integration over all beam directions gives the occluded illuminance. The final result is interpreted in terms of rays cast from the illuminated point. It is found that a system of surfaces establishes a unique set of rays and those, and only those, that strike the source need be considered. Each such intersection produces an increment to the illuminance, calculated from a function that results from the integration over all beams.

Defining flux beams from homogenous diffuse emitters

A differential element, dA, of a homogeneous diffuse emitter, A, produces a differential illuminance, dE(x,y) given by

$$dE(x,y) = L\, d\omega \cos(\xi)\,, \tag{1}$$

where $d\omega = \frac{dA \cos(\theta)}{r^2}$ and L is the diffuse luminance of A. Thus, dE(x,y) can be written as

$$dE(x,y) = L \frac{dA \cos(\theta)\cos(\xi)}{r^2}$$

Vector **R** defines the orientation of solid angle $d\omega$. See Fig. 1. For fixed **R**, the illuminance at any element dA' is given by Eq. 1 if the solid angle $d\omega$ projected from it in direction **R** intersects the emitter A. The illuminance is zero, otherwise.

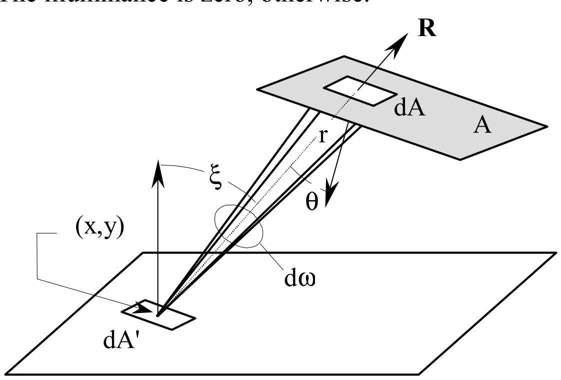

Figure 1. Differential illuminance from an element of diffuse emitter.

In an entirely equivalent way, the illuminance at dA' can be determined from

$$dE(x,y) = \frac{d\phi}{dA'}\,, \tag{2}$$

where $d\phi = I\, d\omega' = I_n \cos(\theta)\, d\omega' = I_n \cos(\theta) \dfrac{dA' \cos(\xi)}{r^2} = L\, dA \cos(\theta) \dfrac{dA' \cos(\xi)}{r^2}$; as shown in Fig. 2.

Thus, the differential illuminance is

$$dE(x, y) = L \frac{dA \cos(\theta) \cos(\xi)}{r^2}$$

which is, of course, the same as before.

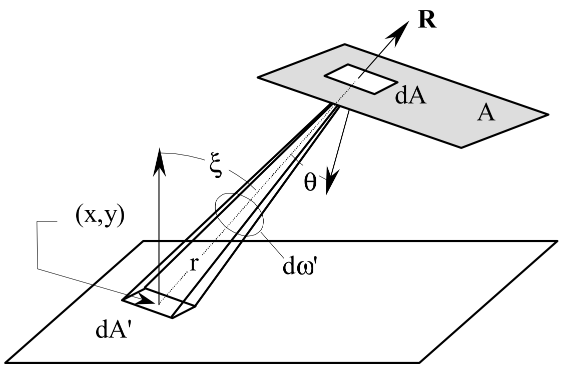

Figure 2. Differential illuminance from an element of diffuse emitter.

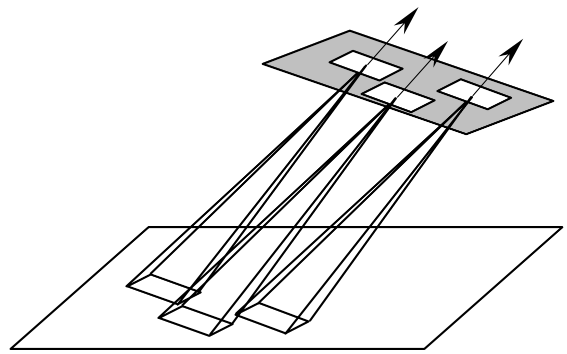

Figure 3. Collective differential solid angles from elements of the emitter.

If the direction, **R** with respect to the plane of the emitting surface is kept constant, and the size of the solid angle dω' fixed, all the elements dA that comprise A, will produce a differential illuminance on the receiving plane, as shown in Fig. 3.

The collection of differential solid angles acts as a beam with direction **R** that defines area A' on the illuminated plane. The beam is incident with angle ξ, has differential divergence of dω', and produces an illuminance given by Eq. 1 anywhere within projected area A'.

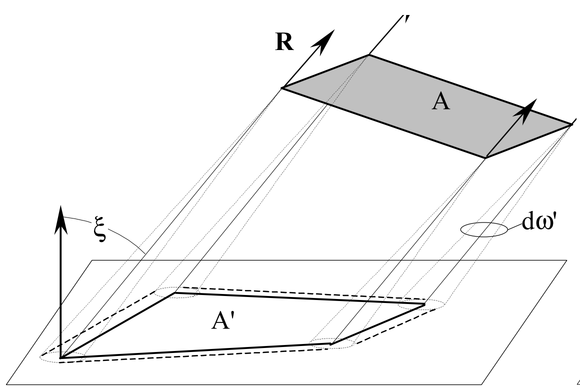

Figure 4. Flux beam geometry.

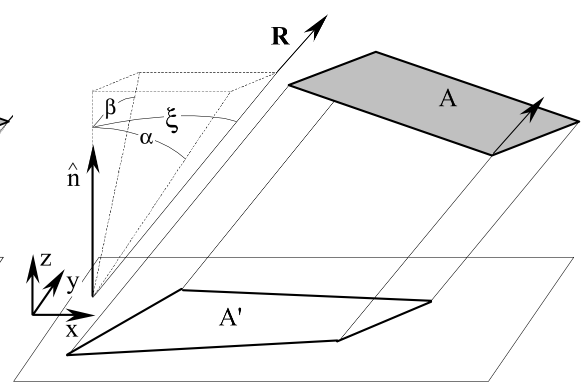

Figure 5. Specification of beam direction by perpendicular-plane angular coordinates.

We establish a Cartesian coordinate system with its z-coordinate axis perpendicular to the illuminated surface. Direction **R** is specified in the perpendicular-plane angular (ppa) coordinate system, angles (α,β) measured from the receiving plane normal, in planes that correspond to the x-z and y-z coordinate planes, respectively. The solid angle that specifies the divergence of the beam, and the cosine of its incident angle are expressed in ppa coordinates by

$$d\omega' = \frac{d\tan(\alpha)\, d\tan(\beta)}{(1 + \tan^2(\alpha) + \tan^2(\beta))^{3/2}} \quad \text{and} \quad \cos(\xi) = \frac{1}{(1 + \tan^2(\alpha) + \tan^2(\beta))^{1/2}} .$$

Thus, the differential illuminance *anywhere within projection A'* is given by

$$dE = L\frac{d\tan(\alpha)\,d\tan(\beta)}{(1+\tan^2(\alpha)+\tan^2(\beta))^2}\ . \tag{3}$$

We now posit a switch function, $S_A(x,y;\tan(\alpha),\tan(\beta))$, the assumes values 1or 0, depending on whether point (x,y) is inside or outside the projection of a beam from A incident from direction (α,β); i.e. inside or outside A'. It proves convenient to have the switch function dependent on the tangent of the ppa angles. The differential illuminance anywhere on the receiving plane is thus

$$dE(x,y) = L\frac{d\tan(\alpha)\,d\tan(\beta)}{(1+\tan^2(\alpha)+\tan^2(\beta))^2} S_A(x,y;\tan(\alpha),\tan(\beta))\ . \tag{4}$$

The total illuminance at any point on the receiving plane will be the cumulative effect of all the beams projected from A that intersect that point. This is equivalent to letting α and β range from -π/2 to +π/2; and so the tangents range from -∞ to +∞. Thus, the total illuminance is obtained by integrating Eq. 4.

$$E(x,y) = L\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_A(x,y;\tan(\alpha),\tan(\beta))\frac{1}{(1+\tan^2(\alpha)+\tan^2(\beta))^2}\,d\tan(\alpha)\,d\tan(\beta)\ . \tag{5}$$

Note that depending on the orientation of the emitter, the range of α and β can produce beams emanating from its backside. We want the illuminance to be zero anywhere in this back-projected beam. Additionally, if it is anticipated that these integrations are to be performed by residue calculus, then restrictions are imposed on the switch function. Namely, it must be everywhere analytic except for isolated singular points—which appears to eliminate any function that exhibits the discontinuity of a switch function. We use an interesting artifice, described below, to avoid this difficulty. To simplify the notation in the rest of this development, we let α represent the tangent of the angle α, and β the tangent of angle β; i.e. (tan(α),tan(β))→(α,β).

Step function defining the intersection of a flux beam and an illuminated plane

The emitter is assumed to have a convex polygonal outline, and thus its projection A' will always be convex. A switch function is constructed from cross products of vectors formed by the illuminated point (x,y) and the vertices of the projection (x'_n,y'_n).

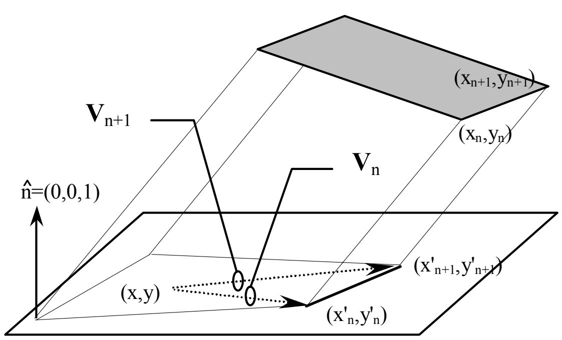

Figure 6. Vectors used in the switch function.

If the coordinates of the vertices of emitter A are always defined with the same handedness, and the normal to the illuminated plane is (0,0,1), then the dot product of the surface normal with the cross product of $\mathbf{V}_n$ into $\mathbf{V}_{n+1}$, will be positive if point (x, y) is on the interior side of the edge formed by (x'_n, y'_n) and (x'_{n+1}, y'_{n+1}). That is,

$$\mathbf{V}_n \times \mathbf{V}_{n+1} \circ \hat{n} = (x'_n - x)(y'_{n+1} - y) - (x'_{n+1} - x)(y'_n - y)$$

The projected vertices are given by

$x'_n = x_n + z_n\,\alpha$,
$y'_n = y_n + z_n\,\beta$.

If these are substituted into the cross product and like terms collected, the result is

$$\mathbf{V}_n \times \mathbf{V}_{n+1} \circ \hat{n} = a_n\alpha + b_n\beta + c_n$$

where $a_n = y\,(-z_n + z_{n+1}) + y_{n+1}\,z_n - y_n\,z_{n+1}$,
$b_n = x\,(-z_n + z_{n+1}) - x_{n+1}\,z_n + x_n\,z_{n+1}$,
$c_n = -y\,(-x_n + x_{n+1}) + x(y_n - y_{n+1}) - x_{n+1}\,y_n - x_n\,y_{n+1}$.

All coordinates are with respect to an origin on the receiving plane. We note that a_n, b_n, and c_n depend only on the position of the illuminated point and the fixed vertices of the n^{th} edge of the emitting polygon—not on the beam direction (α,β). Any cross product, formed by the vertices of an edge of projection A', and (x,y), will be positive if (x, y) is interior to the edge, and negative if (x, y) is exterior.

A unit step function, U, is now posited that transforms a cross product, $a_n\,\alpha + b_n\,\beta + c_n$, into 1 or 0, depending on its *sign*. A little thought shows that the required switch function will be the product of these unit step functions, each evaluated using the cross product formed with consecutive pairs of vertices of the projection of the emitting polygon:

$$S_A(x, y; \alpha, \beta) = \prod_{n=1}^{N} U(a_n\alpha + b_n\beta + c_n) = \begin{cases} 1 \\ 0 \end{cases} \text{if (x,y) is } \begin{matrix}\text{inside} \\ \text{outside}\end{matrix} \text{ the projection of polygon A,}$$

where N = number of vertices of A. Remarkably, with $S_A(x,y;\alpha,\beta)$ defined this way, it is *everywhere* = 0 if the direction (α,β) specifies a beam that emanates from the backside of A, as defined by the order in which its vertices are specified. With a careful drawing and attention to handedness, it can be shown that any point (x,y) is always outside of at least one edge of a projection from the backside of a polygon.

The unit step function is generated as follows. Recognizing that the residue calculus evaluation of Eq. 5 requires analytic functions, we use the integral of an analytic, continuous function to generate a discontinuous function:

$$\int_0^\infty \frac{Z}{k^2 + Z^2}\,dk = \frac{1}{2}\hat{i}\,(\mathrm{Log}(-\frac{\hat{i}}{Z}) - \mathrm{Log}(\frac{\hat{i}}{Z})) = \begin{cases} +\pi/2 \\ -\pi/2 \end{cases} \text{depending on whether Real(Z) is } \begin{cases} > 0 \\ < 0 \end{cases},$$

where $\hat{i} = \sqrt{-1}$. The principle branch of Log is used, cut along the negative real axis. To produce 1 or 0 from this integral, we add

$$\int_0^\infty \frac{1}{k^2 + 1}\,dk = \pi/2\,.$$

These two integrands are added to give:

$$\frac{Z}{k^2 + Z^2} + \frac{1}{k^2 + 1} = \frac{(1+Z)(k^2 + Z)}{(1+k^2)(k^2 + Z^2)}\,,$$

and so the step function is

$$U(Z) = \frac{1}{\pi}\int_0^\infty \frac{(1+Z)(k^2 + Z)}{(1+k^2)(k^2 + Z^2)}\,dk = \begin{cases} 1 + i\,0 \\ 0 + i\,0 \end{cases} \text{if Real(Z) is } \begin{matrix} > 0 \\ < 0 \end{matrix}\,. \qquad (6)$$

Fig. 7 shows this unit step function evaluated for various values of complex argument Z.

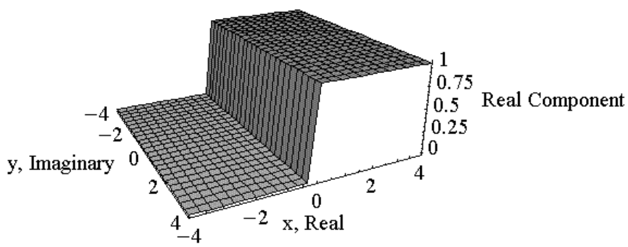

Figure 7. Unit step function defined in Eq. 6.

The reason for these elaborate machinations is that the *integrand* of this integral is everywhere analytic except at isolated singular points. This unit step function has an additional important property that facilitates the necessary integration: the result of the integration is 0 or 1, regardless of the imaginary component of the argument Z—i.e. only the sign of the *real* part of Z affects the result. The switch function is the product of integrals of the type in Eq. 6, one for each edge of emitter A, and its final form is:

$$S_A(\alpha,\beta;x,y)=\prod_{n=1}^{N}\frac{1}{\pi}\int_0^{\infty}\frac{(1+(a_n\alpha+b_n\beta+c_n))(k_n{}^2+(a_n\alpha+b_n\beta+c_n))}{(1+k_n{}^2)(k_n{}^2+(a_n\alpha+b_n\beta+c_n)^2)}dk_n$$

Substituting this into Eq. 5, gives for the illuminance at point (x,y):

$$E(x,y)=L\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{n=1}^{N}\frac{1}{\pi}\int_0^{\infty}\frac{(1+(a_n\alpha+b_n\beta+c_n))(k_n{}^2+(a_n\alpha+b_n\beta+c_n))}{(1+k_n{}^2)(k_n{}^2+(a_n\alpha+b_n\beta+c_n)^2)}dk_n\right)\frac{1}{(1+\alpha^2+\beta^2)^2}d\alpha\, d\beta\ .$$

The integrand is sufficiently well behaved so that the order of integrations can be changed:

$$E(x,y)=L\int_0^{\infty}dk_1\int_0^{\infty}dk_2\cdots\int_0^{\infty}dk_N\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{n=1}^{N}\frac{1}{\pi}\frac{(1+(a_n\alpha+b_n\beta+c_n))(k_n{}^2+(a_n\alpha+b_n\beta+c_n))}{(1+k_n{}^2)(k_n{}^2+(a_n\alpha+b_n\beta+c_n)^2)}\right)\frac{1}{(1+\alpha^2+\beta^2)^2}d\alpha\, d\beta$$

One additional simplification is necessary: the algebraic term that expresses the purely geometric function $d\omega\cos(\xi)$, must be simplified. More specifically, α and β must be separated. The function

$$\frac{1}{(1+\alpha^2+\beta^2)^2}$$

has the form shown in Fig. 8.

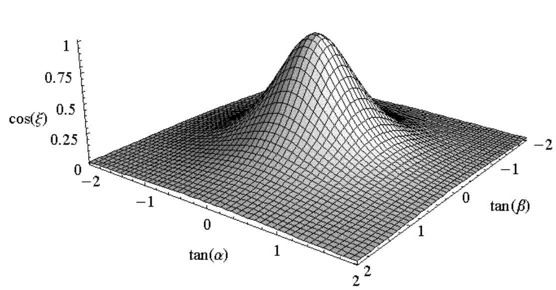

Figure 8. Geometric function of α and β.

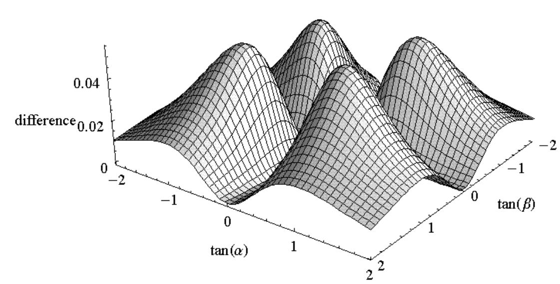

Figure 9. Error in the approximation to the geometry function.

This can be approximated by

$$\frac{1}{(1+\alpha^2+\beta^2)^2} \approx \frac{1}{(1+\alpha^2)^2}\frac{1}{(1+\beta^2)^2}$$

and each of the latter elements, in turn, approximated by

$$\frac{1}{(1+\alpha^2)^2} \approx \left(\frac{\hat{i}}{\hat{i}\,G+H\alpha}\right)^2 + \left(\frac{\hat{i}}{-\hat{i}\,G+H\alpha}\right)^2 + \ldots$$

The approximation can involve as many pairs of terms as necessary. It has been found sufficient to use two pair, i.e. four terms. The constants G and H are fixed once and for all. The approximation for the geometry function is thus,

$$\frac{1}{(1+\alpha^2+\beta^2)^2} \approx \sum_{p=1}^{4}\sum_{q=1}^{4}\left(\frac{\hat{i}}{\hat{i}\,G_p+H_p\alpha}\right)^2\left(\frac{\hat{i}}{\hat{i}\,G_q+H_q\beta}\right)^2$$

where $G_1 = 1.57$, $G_2 = 3.25$, $G_3 = -1.57$, $G_4 = -3.25$
$H_1 = 1.32$, $H_2 = .525$, $H_3 = 1.32$, $H_4 = .525$

This approximation differs from the correct geometry function by no more that 0.04, as shown in Fig. 9. The final form for the integral expressing the illuminance is

$$E(x,y) = L\int_0^\infty dk_1 \int_0^\infty dk_2 \cdots \int_0^\infty dk_N \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{n=1}^{N}\frac{1}{\pi}\frac{(1+(a_n\alpha+b_n\beta+c_n))(k_n{}^2+(a_n\alpha+b_n\beta+c_n))}{(1+k_n{}^2)(k_n{}^2+(a_n\alpha+b_n\beta+c_n)^2)}\right)$$
$$\sum_{p=1}^{4}\sum_{q=1}^{4}\left(\frac{\hat{i}}{\hat{i}\,G_p+H_p\alpha}\right)^2\left(\frac{\hat{i}}{\hat{i}\,G_q+H_q\beta}\right)^2 d\alpha\, d\beta \qquad (7)$$

Illuminance at a point from integration over all beams

Residue calculus is used to perform the integrations over α, β, and finally over the k_n.[9] The details are in the Appendix. The result can be expressed as follows. Define the quantities

$$\alpha_{m,n} = \frac{b_n c_m - b_m c_n}{a_n b_m - a_m b_n} \qquad (8a)$$

$$\beta_{m,n} = \frac{a_n c_m - a_m c_n}{-a_n b_m + a_m b_n} \qquad (8b)$$

$$A_m = \frac{a_m}{c_m} \qquad (8c)$$

$$B_m = \frac{b_m}{c_m} \qquad (8d)$$

and the function $D(\alpha_{m,n}, \beta_{m,n}: A_m, B_m)$:

$$D(\alpha_{m,n},\beta_{m,n};A_m,B_m) \;=\; \mathrm{sign}(A_nB_m - A_mB_n)\,A_m \sum_{p=1}^{4}\sum_{q=1}^{4}$$

$$\mathrm{Log}\left(\frac{H_qA_m\left(\hat{i}\,G_p + H_p\alpha_{m,n}\right)}{H_pB_m\left(\hat{i}\,G_q + H_q\beta_{m,n}\right)}\right)\left[\frac{B_m}{\left((G_pH_qA_m + G_qH_pB_m) + \hat{i}\,H_pH_q\right)^2} + \frac{1}{H_p\left(G_q + \hat{i}\,H_q\beta_{m,n}\right)\left(G_pH_qA_m + G_qH_pB_m + \hat{i}\,H_pH_q\right)^2}\right]$$

(8e)

The logarithm must be evaluated with care. Finally, the illuminance at point (x, y) is

$$E(x,y) = \sum_{m=1}^{N}\;\sum_{n=1,\neq m}^{N}\left(\prod_{j=1\neq m,n}^{N} U(a_j\alpha_{m,n} + b_j\beta_{m,n} + c_j)\right) D(\alpha_{m,n},\beta_{m,n};A_m,B_m) \tag{9}$$

Geometric Interpretation

Application of this result, and its extension to the case that includes occluding objects, is facilitated by a geometric interpretation. The quantities defined above, a_n, b_n, c_n, are the x, y, and z components, respectively, of the cross product of the two vectors formed by the illuminated point, (x, y) and the vertices of the n^{th} edge of the polygon. We will refer to the plane that contains these two vectors as an *edge-plane*. The quantities a_n, b_n, c_n, are, therefore, the components of a vector, $\mathbf{P}_n$, perpendicular to the edge-plane, with magnitude equal to twice the area of the surface formed by (x,y) and the edge having coordinates (x_n, y_n)-(x_{n+1}, y_{n+1}). This is shown in Fig. 10.

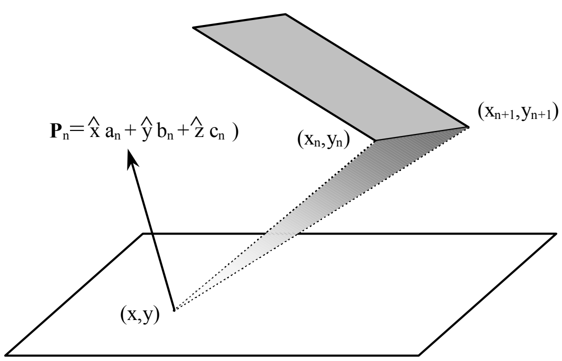

Figure 10. Formation of edge-plane and the vector $\mathbf{P}_n$ perpendicular to it.

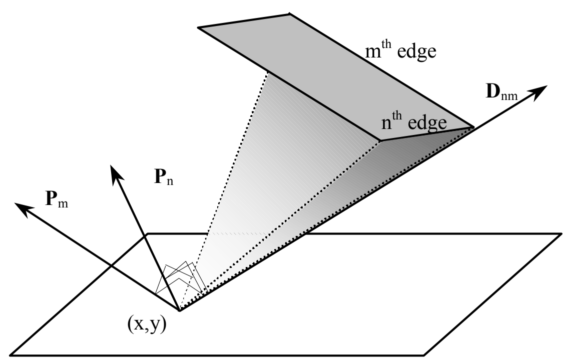

Figure 11. Formation of vector D_{mn} from the intersection of two edge planes.

The quantities A_m and B_m are the tangents, $\tan(\alpha_m)$, $\tan(\beta_m)$, of the ppa angles specifying the direction of the m^{th} vertex of the polygon from the illuminated point (x,y). As shown in Eqs. 8, $\alpha_{m,n}$ and $\beta_{m,n}$ are composed of the quantities

$$b_nc_m - b_mc_n\,,$$
$$a_nc_m - a_mc_n\,,$$
$$a_nb_m - a_mb_n\,,$$

and it can be seen that these are proportional to the x, y, and z-components respectively, of a vector $\mathbf{D}_{nm}$, formed by the cross product of two edge-plane normal vectors, $\mathbf{P}_m$ and $\mathbf{P}_n$; i.e.,

$$\mathbf{D}_{nm} = \mathbf{P}_n \times \mathbf{P}_m.$$

Such a vector defines the direction of the intersection of the two edge-planes specified by (x,y) and the normal vectors $\mathbf{P}_m$ and $\mathbf{P}_n$, as shown in Fig. 11. That is

$$D_{nm,x} = b_n c_m - b_m c_n$$
$$D_{nm,y} = -(a_n c_m - a_m c_n)$$
$$D_{nm,z} = a_n b_m - a_m b_n$$

Substituting into Eqs. 8a and 8b gives:

$$\alpha_{m,n} = \frac{b_n c_m - b_m c_n}{a_n b_m - a_m b_n} = \frac{D_{nm,x}}{D_{nm,z}} = \tan(\alpha_{D\,nm}) \qquad (10a)$$

$$\beta_{m,n} = \frac{a_n c_m - a_m c_n}{-a_n b_m + a_m b_n} = \frac{-D_{nm,y}}{-D_{nm,z}} = \tan(\beta_{D\,nm}) \qquad (10b)$$

And so α_{mn} β_{mn} are the tangents of the ppa angles specifying the direction of $\mathbf{D}_{nm}$. Therefore, the unit step function in Eq. 9, $U(a_j\, \alpha_{mn} + b_j\, \beta_{mn} + c_j)$, returns 1 or 0, depending on whether point (x, y) is contained within the projection at tangents (α_{mn}, β_{mn}) of the edges j=1,…N; $j \neq m,n$, onto the receiving plane. Or, what is the same thing, *whether a ray cast from point (x, y) in direction specified by* $\mathbf{D}_{nm}$*, intersects the surface bounded by edges j=1,…N;* $j \neq m,n$. Contact with the surface at its edges is considered an intersection. Contact on the backside of a surface is *not* considered an intersection. Note that the polygon to be intersected is missing the edges that are used to generate $\mathbf{D}_{nm}$. A little thought shows that the polygon to be intersected can be defined *including* them and the result will be the same. Though less obvious, this is true even if the missing edges are not adjacent. Additionally, considering the pair of edges (n,m) in reverse order, (m,n), reverses the direction of the D-vector, i.e. $\mathbf{D}_{mn} = -\mathbf{D}_{nm}$, but produces the same tangents $\alpha_{n,m}$ and $\beta_{n,m}$. A different value of the D-function results, however.

Equation 9 can be interpreted as a process that generates rays from the intersection of all possible pairs of edge-planes and if such a ray intersects the emitting polygon, the illuminance is incremented by the appropriately evaluated D-function:

$$E(x,y) = \sum_{m=1}^{N} \sum_{n=1,\neq m}^{N} I_A(x,y;\alpha_{m,n},\beta_{m,n})\; D(\alpha_{m,n},\beta_{m,n};A_m,B_m), \qquad (11)$$

where $I_A(x,y;\alpha_{mn},\beta_{mn}) = 1$ or 0, depending on whether the ray from (x,y), projected in direction (α_{mn}, β_{mn}), intersects polygon A, or not. An example is shown in Fig. 12. A luminous triangle is positioned above an illuminated plane, oriented so that its inactive side is seen by part of the illuminated plane and therefore has no illuminance. The illuminance calculated with the new procedure is shown in Fig. 13a. A comparison with values calculated using configuration factors is shown in Fig. 13b. Illuminance is calculated along the dashed line under the luminous triangle in the illuminated plane of Fig. 12. The larger values are within ±2% of those calculated with configuration factors.

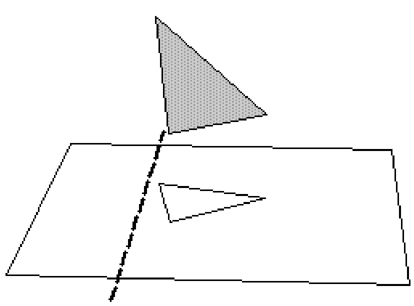

Figure 12. Simple test case: luminous triangle tipped with respect to illuminated plane.

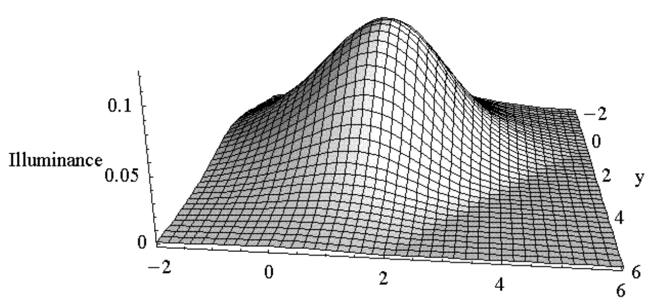

Figure 13a. Illuminance due to tipped triangle.

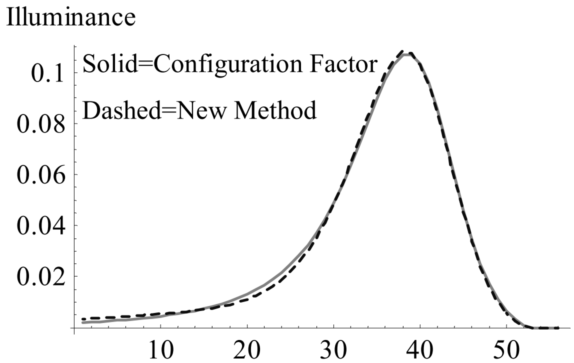

Figure 13b. New and traditional illuminance calculations compared.

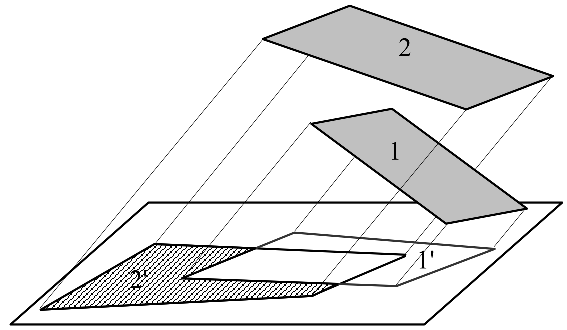

Figure 14. Projection of emitting surface with occluding surface on the illuminated plane.

Blocking beams

This basic procedure is now extended to account for occluding surfaces. To establish the principle involved, it is sufficient to consider the case of one occluding surface. The arrangement is shown in Fig. 14. As before, a differentially divergent beam of flux emanates from surface 2 and produces a uniform differential illuminance anywhere within its projection. The beam is occluded by surface 1, so that the projection of 2 that produces illuminance is only the shaded section. Using the switch function described above, a point (x,y) will be illuminated if

$$(1 - S_1(x, y; \alpha, \beta))S_2(x, y; \alpha, \beta) = 1\,,$$

where subtracting the switch function from 1 reverses its effect, and produces 1 if the point is *outside* the projection, and zero otherwise. Thus, (x,y) is illuminated by the beam from direction (α,β) if it is *inside* projection 2', and *outside* projection 1'. The illuminance due to all the beams, and therefore due to emitter polygon 2, occluded by polygon 1 is:

$$E(x, y) = L \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1 - S_1(x, y; \alpha, \beta))S_2(x, y; \alpha, \beta) \sum_{p=1}^{4}\sum_{q=1}^{4} \left(\frac{\hat{i}}{\hat{i}\, G_p + H_p\alpha}\right)^2 \left(\frac{\hat{i}}{\hat{i}\, G_q + H_q\beta}\right)^2 d\alpha\, d\beta\ .$$

Where the approximation for the geometry function is used as before.

Now assume a system of M total surfaces. Let them be numbered such that the last surface is the emitter. Assume further that none of the other M-1 surfaces are entirely behind the emitting surface or the illuminated plane. That is, no other surfaces have the possibility of occluding.[10] The effect of any occluding polygon is obtained by multiplying by one minus its switch function. The illuminance at a point (x,y) on the illuminated surface is given by

$$E(x, y) = L \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\prod_{m=1}^{M-1} (1 - S_m(x, y; \alpha, \beta))\right) S_M(x, y; \alpha, \beta) \sum_{p=1}^{4}\sum_{q=1}^{4} \left(\frac{\hat{i}}{\hat{i}\, G_p + H_p\alpha}\right)^2 \left(\frac{\hat{i}}{\hat{i}\, G_q + H_q\beta}\right)^2 d\alpha\, d\beta\ . \qquad (12)$$

As above, each switch function is defined by the product of unit step functions. We have for the switch function for the projection in direction (α,β) of the mth polygon:

$$S_m(\alpha,\beta;x,y) = \prod_{n=1}^{N_m} \frac{1}{\pi} \int_0^\infty \frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)} dk_{m,n}$$

or

$$S_m(\alpha,\beta;x,y) = \left(\prod_{n=1}^{N_m} \int_0^\infty dk_{m,n}\right) \prod_{n=1}^{N_m} \frac{1}{\pi} \frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}$$

Where N_m = number of edges of the m^{th} polygon, and for the n^{th} edge of the m^{th} polygon:

$a_{m,n} = y(-z_{m,n} + z_{m,n+1}) + y_{m,n+1} z_{m,n} - y_{m,n} z_{m,n+1}$,
$b_{m,n} = x(-z_{m,n} + z_{m,n+1}) - x_{m,n+1} z_{m,n} + x_{m,n} z_{m,n+1}$,
$c_{m,n} = -y(-x_{m,n} + x_{m,n+1}) + x(y_{m,n} - y_{m,n+1}) - x_{m,n+1} y_{m,n} - x_{m,n} y_{m,n+1}$.

If these switch functions are substituted into Eq. (12), we obtain for the illuminance at a point (x,y)

$$E(x,y) = L\left(\prod_{m=1}^{M}\prod_{n=1}^{N_m}\int_0^\infty dk_{m,n}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{m=1}^{M-1}\left(1-\left(\prod_{n=1}^{N_m}\frac{1}{\pi}\frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}\right)\right)\right)$$
$$\left(\prod_{n=1}^{N_M}\frac{1}{\pi}\frac{(1+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n}))(k_{M,n}{}^2+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n}))}{(1+k_{M,n}{}^2)(k_{M,n}{}^2+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n})^2)}\right)$$
$$\sum_{p=1}^{4}\sum_{q=1}^{4}\left(\frac{\hat{i}}{\hat{i}\,G_p+H_p\alpha}\right)^2\left(\frac{\hat{i}}{\hat{i}\,G_q+H_q\beta}\right)^2 d\alpha\, d\beta \qquad (13)$$

Residue calculus is again used to perform the integrations. The result is:

$$E(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N_m}\sum_{p=1}^{M}\sum_{q=1;(p,q)\neq(m,n)}^{N_p} I_M(x,y;\alpha_{m,n;p,q},\beta_{m,n;p,q})\, I_m(x,y;\alpha_{m,n;p,q},\beta_{m,n;p,q})\, I_p(x,y;\alpha_{m,n;p,q},\beta_{m,n;p,q})$$
$$\prod_{k=1\neq m\neq n}^{M-1}(1+I_k(x,y;\alpha_{m,n;p,q},\beta_{m,n;p,q}))\, D(\alpha_{m,n;p,q},\beta_{m,n;p,q};A_{m.n},B_{m.n}) \qquad (14)$$

where,

$$I_k(x,y;\alpha_{m,n;p,q},\beta_{m,n;p,q}) = -1^{1+[k/M]} \text{ or } 0,$$

depending on whether a ray projected from (x,y) in direction ($\alpha_{m,n;p,q}$, $\beta_{m,n;p,q}$), intersects the k^{th} polygon, or not. The D-function is the same as that described above. The brackets, [ ], indicate integer division.

A geometric interpretation of the result is as follows. Rays are formed by the intersection of all possible pairs of edge-planes; e.g., the edge-plane of the n^{th} edge of the m^{th} polygon, intersecting the edge-plane of the q^{th} edge of the p^{th} polygon. The rays are cast from point (x,y) in the direction $\alpha_{m,n;p,q}$, $\beta_{m,n;p,q}$, formed by these intersections. Examination of Eq. 14 shows that the following obtains:

a) a ray generated by an edge-plane of an occluding surface must intersect that surface,
b) a ray *not* generated by any edge-plane of an occluding surface must *not* intersect that surface, and
c) a ray must intersect the emitting polygon.

If a ray does not satisfy all these conditions, it is not counted. For each ray that does satisfy these conditions, the D-function is evaluated and the illuminance is incremented by that amount. The number of rays that count, even in a complicated system, is small and so is the number of evaluations of the D-function.

Figures 15, 16 and 17 show examples of rays formed by pairs of edge planes.

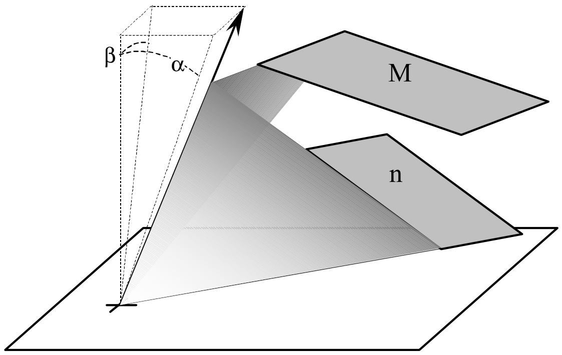

Figure 15. Edge-plane ray that does not count.

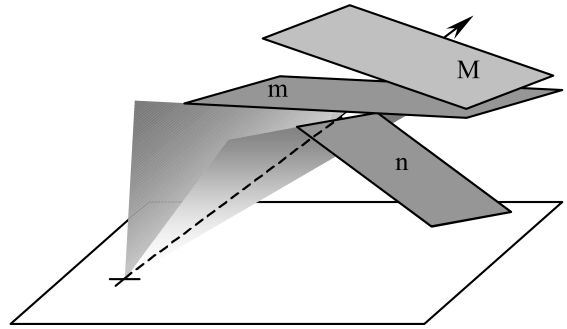

Figure 16. Edge-plane ray that counts.

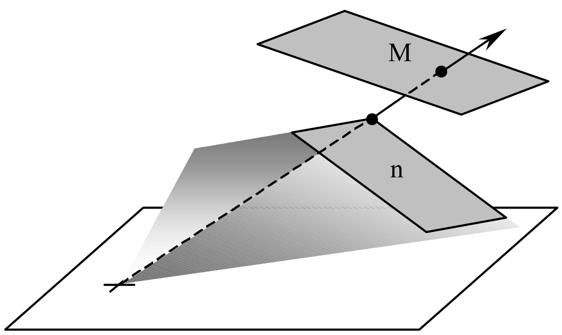

Figure 17. Edge-plane ray that counts.

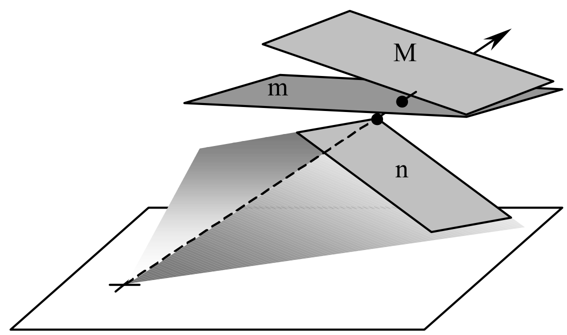

Figure 18. Edge-plane ray that does not count.

An example of occluding polygons is shown in Figure 19. The same luminous triangle is used as in the previous example. Two others are added, shown in darker gray, positioned between the luminous triangle and the illuminated plane. Their horizontal projections are shown in the figure. Figure 20 shows the result of using Eq. 14 to calculate the illuminances. The number of evaluations of the D-function at each illuminated point ranged from zero to 12, with the average being approximately 7.

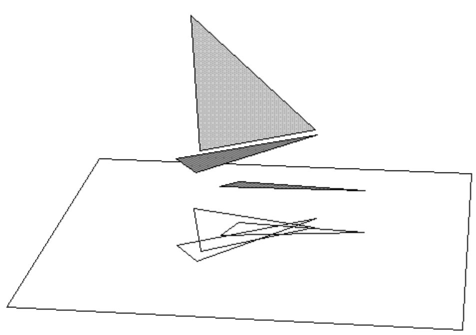

Figure 19. Emitting polygon and two occluding polygons.

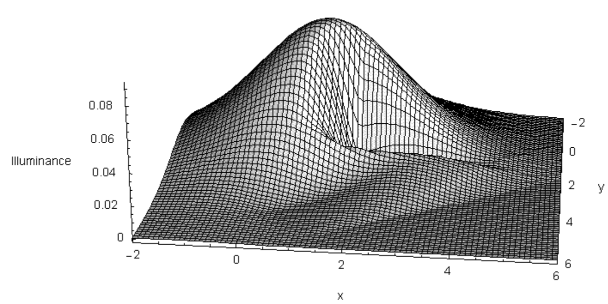

Figure 20. Occluded illuminances.

Discussion

Any of the powerful culling procedures developed for ray casting in computer graphics can be used to isolate only those rays that might intersect the emitting polygon—though the new procedure invites culling based on the orientation of edge-planes, which may prove more efficient. Additionally, ray intersection algorithms have an important place here. One can interpret the new procedure as *casting weighted rays*, with the D-function being the weight. The new procedure developed here establishes an interesting and unexpected nexus between two analysis procedures: traditional radiative transfer and ray casting.

The new procedure lends itself to the efficient analysis of a system of surfaces. One begins with the surface closest to the illuminated plane, treating that surface as an emitter. The next closest surface is then becomes the emitter, with the first surface acting an occluding surface, and so on.

Future research will focus on finding a way to integrate the final equations obtained here over a receiving surface, in order to obtain the total flux received; or what is the same thing, the radiative exchange form factor in the presence of occluding surfaces.

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[8] Seider, C., "A New Method for Calculating Occluded Configuration Factors in Radiative Transfer Analysis," Thesis, University of Colorado at Boulder, 2000.
[9] Ablowitz, M.J. and A. S. Fokas, *Complex Variables: Introduction and Applications*, Cambridge, 1997, pp. 206-245.
[10] Use of a binary space partition tree for cataloguing the polygons in a system is one way in which this can be arranged. Another, and simpler, is to using a bounding box that contains the emitting and receiving polygon and an polygons intersecting the box. See Ashdown, *Radiosity: A Programmers Perspective*, Wiley, 1994, p. 386.

Appendix

The most general form of the integrations to be performed in this development is that in Eq. 13. The condition without occluding surfaces (Eq. 7) is a special case.

$$L\prod_{m=1}^{M}\prod_{n=1}^{N_m}\int_0^{\infty}dk_{m,n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{m=1}^{M-1}\left(1-\left(\prod_{n=1}^{N_m}\frac{1}{\pi}\frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}\right)\right)\right)$$
$$\left(\prod_{n=1}^{N_M}\frac{1}{\pi}\frac{(1+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n}))(k_{M,n}{}^2+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n}))}{(1+k_{M,n}{}^2)(k_{M,n}{}^2+(a_{M,n}\alpha+b_{M,n}\beta+c_{M,n})^2)}\right)$$
$$\sum_{p=1}^{4}\sum_{q=1}^{4}\left(\frac{\hat{i}}{\hat{i}\,G_p+H_p\alpha}\right)^2\left(\frac{\hat{i}}{\hat{i}\,G_q+H_q\beta}\right)^2 d\alpha\, d\beta$$

If this is expanded, one obtains several terms, all of which are of this form:

$$\left(\prod_{m=1}^{M}\prod_{n=1}^{N_m}\int_0^{\infty}dk_{m,n}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\prod_{m=1}^{M}\prod_{n=1}^{N_M}\frac{1}{\pi}\frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}\right).$$

$$\sum_{p=1}^{4}\sum_{q=1}^{4}\left(\frac{\hat{i}}{\hat{i}\,G_p+H_p\alpha}\right)^2\left(\frac{\hat{i}}{\hat{i}\,G_q+H_q\beta}\right)^2 d\alpha\, d\beta \tag{A0}$$

The integrals over α and β are performed using residue calculus. In each case, the closed path of integration is the real number line and a semicircle in the upper half of the complex plane. The value of the integrand vanishes along the semicircle, so the integral along the real number line is equal to $2\pi i$ times the sum of all the residues at the poles in α and β that the integrand has in the upper half-plane. All the poles are simple and are located where denominators vanish. Poles due to the geometry function are located at

$$\alpha=-\hat{i}\frac{G_p}{H_p}.$$

Since H_p is always positive, this is an upper half-plane pole when G_p is negative. Now, the integrand in Eq. A0 can be considered the product of a function in α, $\Phi(\alpha)$, and the denominator of the α-component of the geometry function:

$$\frac{\Phi(\alpha)}{(\hat{i}\,G_p+H_p\,\alpha)^2}.$$

The residue of this at the pole is, since the denominator is squared (See Ablowitz, p. 209):

$$\left.\frac{\partial\Phi(\alpha)}{\partial\alpha}\right|_{\alpha=-\hat{i}\frac{G_p}{H_p}}.$$

But $\Phi(\alpha)$ is the product of switch functions, which have derivatives equal to zero. This is true even though the pole introduces an imaginary component into their argument. Thus, the residues at poles due to the geometry function are zero.

All other poles in α in Eq. A0 are from that portion of switch function denominators that can vanish:

$$\frac{1}{(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}.$$

Its poles are

$$\alpha=\frac{-\beta\, b_{m,n}-c_{m,n}\pm\hat{i}\,k_{m,n}}{a_{m,n}}.$$

Since $a_{m,n}$ may be negative, the pole in the upper half-plane may be expressed as:

$$\alpha=-\frac{\beta\, b_{m,n}+c_{m,n}}{a_{m,n}}+\hat{i}\frac{k_{m,n}}{\sqrt{a_{m,n}^2}}.$$

As before, we consider the integrand to be the product of a function in α, $\Phi(\alpha)$, and this denominator.

$$\frac{\Phi(\alpha)}{(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n})^2)}$$

Since the power of the denominator is 1, there is no differentiation of this function and the residue is simply:

$$\Phi(\alpha)\Big|_{\alpha=\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}+\hat{i}\frac{k_{m,n}}{\sqrt{a_{m,n}^2}}}$$

In this case, $\Phi(\alpha)$ consists of: the α-component of the geometry function, the product of all other switch functions, and the current switch function without part of its denominator. The latter is:

$$\frac{(1+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))(k_{m,n}{}^2+(a_{m,n}\alpha+b_{m,n}\beta+c_{m,n}))}{(1+k_{m,n}{}^2)}.$$

If this is evaluated at the pole, the result is $\frac{1}{2\pi a_{m,n}}$. Thus, this portion of $\Phi(\alpha)$ becomes a constant when evaluated at the pole. The second part of $\Phi(\alpha)$, the product of all other switch functions, is evaluated at the pole. In all these cases it is sufficient to use only the real part of the pole location,

$$\alpha=-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}},$$

since a switch function's value is independent of any imaginary component of its argument. Thus, for example, the switch function for the $(p,q)^{th}$ element of $\Phi(\alpha)$ becomes:

$$\left(\frac{1}{\pi}\frac{(1+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))(k_{p,q}{}^2+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))}{(1+k_{p,q}{}^2)(k_{p,q}{}^2+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q})^2)}\right) \quad \text{(A1)}$$

The first part of $\Phi(\alpha)$, the α-component of the geometry function, evaluated at this pole is:

$$\left(\frac{i}{H_p-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}+\hat{i}\left(\frac{k_{m,n}}{\sqrt{a_{m,n}^2}}+G_p\right)}\right)^2.$$

We note that in the process of evaluating the residues in α, the integration over $k_{m,n}$ from zero to infinity of a switch function, migrates to the integration of the α-component of the geometry function, since the variable of integration, $k_{m,n}$, now appears there. The β-components of the geometry function are not affected by this process.

Integration over α equals $2\pi i$ times the sum of all these residues. Since there are as many residues as there are edges of polygons, integration over α yields as many terms as edges of polygons. Grouping the now-modified α-component and the β-component of the geometry function together, each term of the integration over α has a typical form given by:

$$\frac{2\pi \hat{i}}{2\pi a_{m,n}} \int_{-\infty}^{\infty} \left(\prod_{p=1}^{M} \prod_{q=1}^{N_p} \frac{1}{\pi} \frac{(1+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))(k_{p,q}{}^{2}+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))}{(1+k_{p,q}{}^{2})(k_{p,q}{}^{2}+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q})^{2})} \right)$$

$$\left(\left(H_p - \frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}} + \hat{i}\left(\frac{k_{m,n}}{\sqrt{a_{m,n}^{2}}}+G_p\right)\right)\left(\hat{i}\, G_q + H_q \beta\right)\right)^{-2} d\beta \tag{A2}$$

Equation A2 is now integrated over β. The denominator of the geometry function is squared and, since the rest of the function is the product of switch functions with derivative of zero, the residues of at the poles in β of the geometry function are zero.

The part of a typical component of a switch functions in Eq. A2 with a denominator that can vanish is

$$\frac{1}{(k_{p,q}{}^{2}+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q})^{2})}.$$

Its poles are at

$$\beta = -\frac{a_{p,q}c_{m,n}+a_{m,n}c_{p,q}}{-a_{p,q}b_{m,n}+a_{m,n}b_{p,q}} \pm \hat{i}\,\frac{a_{m,n}k_{p,q}}{-a_{p,q}b_{m,n}+a_{m,n}b_{p,q}}$$

Which pole is in the upper half-plane depends on the sign of $a_{m,n}/(-a_{p,q}b_{m,n}+a_{m,n}b_{p,q})$. The residues are obtained by substituting this value of β into the remaining portions of Eq. A2. Substituting into the remaining component of the switch function that has this pole,

$$\frac{(1+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))(k_{p,q}{}^{2}+(a_{p,q}\left(-\frac{\beta b_{m,n}+c_{m,n}}{a_{m,n}}\right)+b_{p,q}\beta+c_{p,q}))}{(1+k_{p,q}{}^{2})},$$

gives

$$\frac{1}{2\pi}\frac{a_{m,n}}{(-a_{p,q}b_{m,n}+a_{m,n}b_{p,q})}.$$

Only the real part of the pole value of β need be substituted into the other switch function components, giving, for the s^{th} component of the r^{th} switch function:

$$\frac{(1+(a_{r,s}\alpha_{m,n;p,q}+b_{r,s}\beta_{m,n;p,q}+c_{p,q}))(k_{r,s}{}^{2}+(a_{r,s}\alpha_{m,n;p,q}+b_{r,s}\beta_{m,n;p,q}+c_{r,s}))}{(1+k_{r,s}{}^{2})(k_{r,s}{}^{2}+(a_{r,s}\alpha_{m,n;p,q}+b_{r,s}\beta_{m,n;p,q}+c_{r,s})^{2})}$$

where

$$\alpha_{m,n;p,q} = \frac{b_{p,q}c_{m,n}+b_{m,n}c_{p,q}}{a_{p,q}b_{m,n}-a_{m,n}b_{p,q}} \quad \text{and} \quad \beta_{m,n;p,q} = \frac{a_{p,q}c_{m,n}+a_{m,n}c_{p,q}}{-a_{p,q}b_{m,n}+a_{m,n}b_{p,q}} \tag{A3}$$

Finally, the location of the pole in β is substituted into the geometry function. It now contains $k_{p,q}$, as well as $k_{m,n}$. Integrating each from 0 to ∞ gives, after much simplification, the D-function defined in Eq. 8e.

By this process, if the system of polygons contains a total of M edges, integration over α generates M terms. For each of these, integration over β generates (M-1) terms, for a total of M(M-1) terms, most of which evaluate to zero. It remains to show that this is the case and, in the process, give a geometric interpretation to these results.

This can be made clear by way of example. Consider the case of four polygons, each with four sides. The function to be integrated over α and β has the following structure:

$$S_4(1-S_3)(1-S_2)(1-S_1)\,G\,,$$

where S_n is the switch function for the n[th] polygon, and G is the geometry function. Polygon four is the emitter, and the rest can potentially occlude. If this is expressed in terms of unit step functions, we have

$$U_{4,1}U_{4,2}U_{4,3}U_{4,4}(1-U_{3,1}U_{3,2}U_{3,3}U_{3,4})(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,G\,, \qquad \text{(A4)}$$

where $U_{m,n}$ is the unit step function from the n[th] edge of the m[th] polygon. Integration over α is expressed by the sum of the 16 residues that this function has at the simple poles in the upper half-plane that each of the 16 unit step functions has in α. Consider the residue from $U_{3,2}$. Equation A4 can be written

$$\begin{aligned}&U_{4,1}U_{4,2}U_{4,3}U_{4,4}(1)(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,G+\\&U_{4,1}U_{4,2}U_{4,3}U_{4,4}(-U_{3,1}U_{3,2}U_{3,3}U_{3,4})(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,G\,.\end{aligned}$$

Only the second term has a pole in $U_{3,2}$, and as shown above, the unit step function $U_{3,2}$ becomes a constant. Thus, the residue at the pole in $U_{3,2}$ has a structure of unit step functions that is:

$$U_{4,1}U_{4,2}U_{4,3}U_{4,4}(-U_{3,1}U_{3,3}U_{3,4})(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,G'\,, \quad \text{(A5)}$$

where G' is geometry function modified by the integration over α, and includes the constant from $U_{3,2}$. The switch function S_3 looses the factor of 1, and the sense in which it switches becomes inverted. Note that only the residues at the poles of the unit step functions from the occluding polygons have an altered structure. Thus, integration over α generates 12 terms with a structure as in Eq. A5, and 4 with a structure as in Eq. A4.

Integration over β changes the structure in a similar way. The result is three possible final forms to the $16\times15=240$ residues:

a) Residues at poles in α and β both from unit step functions from the emitter polygon; $U_{4,2}$ and $U_{4,4}$ for example:

$$U_{4,1}U_{4,3}(1-U_{3,1}U_{3,2}U_{3,3}U_{3,4})(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,D$$

This reside will be zero unless the point defined by ($\alpha_{4,2;4,4}$, $\beta_{4,2;4,4}$) is *within* the emitting polygon, now defined by $U_{4,1}$ $U_{4,3}$, and *outside* all the occluding polygons.

b) Residues at a pole in α or β from a unit step function from the emitter polygon, and at a pole from a unit step function from an occluding polygon; $U_{4,3}$ and $U_{2,2}$ for example:

$$U_{4,1}U_{4,2}U_{4,4}(1-U_{3,1}U_{3,2}U_{3,3}U_{3,4})(-U_{2,1}U_{2,3}U_{2,4})(1-U_{1,1}U_{1,2}U_{1,3}U_{1,4})\,D$$

This reside will be zero unless the point defined by ($\alpha_{4,3;2,2}$, $\beta_{4,3;2,2}$) is *within* the emitting polygon, now defined by $U_{4,1}$ $U_{4,2}$ $U_{4,4}$, *within* the occluding polygon defined by $U_{2,1}$ $U_{2,3}$ $U_{2,4}$,and *outside* all other occluding polygons.

c) Residues at poles in α and β both from unit step functions from occluding polygons; $U_{3,2}$ and $U_{1,3}$ for example::

$$U_{4,1}U_{4,2}U_{4,3}U_{4,4}(-U_{3,1}U_{3,3}U_{3,4})(1-U_{2,1}U_{2,2}U_{2,3}U_{2,4})(-U_{1,1}U_{1,2}U_{1,4})\,D$$

This reside will be zero unless the point defined by ($\alpha_{3,2;1,3}$, $\beta_{3,2;1,3}$) is *within* the emitting polygon, *within* the occluding polygons defined by $U_{3,1}$ $U_{3,3}$ $U_{3,4}$ and $U_{1,1}$ $U_{1,2}$ $U_{1,4}$,and *outside* all other occluding polygons.

These three conditions define a geometric interpretation of the final result.

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The D-function has a zero imaginary component. Extracting only the real component of Eq. 8e gives the following. Note that only the positive constants G are used, so the number of summands is halved.

$$D(\alpha_{m,n},\beta_{m,n};A_m,B_m) = \operatorname{sign}(-A_nB_m+A_mB_n)\sum_{p=1}^{2}\sum_{q=1}^{2}$$

$$4\operatorname{atan}\left(\frac{H_p}{G_p}\alpha_{m,n}\right)A_mB_m\left(H_pH_q\left(\frac{G_pH_qA_m-G_qH_pB_m}{\left((G_pH_qA_m-G_qH_pB_m)^2+H_p^2H_q^2\right)^2}+\frac{G_pH_qA_m+G_qH_pB_m}{\left((G_pH_qA_m+G_qH_pB_m)^2+H_p^2H_q^2\right)^2}\right)\right)-$$

$$4\operatorname{atan}\left(\frac{H_q}{G_q}\beta_{m,n}\right)A_mB_m\left(H_pH_q\left(\frac{-G_pH_qA_m+G_qH_pB_m}{\left((G_pH_qA_m-G_qH_pB_m)^2+H_p^2H_q^2\right)^2}+\frac{G_pH_qA_m+G_qH_pB_m}{\left((G_pH_qA_m+G_qH_pB_m)^2+H_p^2H_q^2\right)^2}\right)\right)+$$

$$\operatorname{Log}\left(\frac{G_q^2H_p^2B_m^2\left(1+\frac{H_q^2}{G_q^2}\beta^2{}_{m,n}\right)}{G_p^2H_q^2A_m^2\left(1+\frac{H_p^2}{G_p^2}\alpha^2{}_{m,n}\right)}\right)A_mB_m\frac{(G_pH_qA_m-G_qH_pB_m)^2-H_p^2H_q^2}{\left((G_pH_qA_m-G_qH_pB_m)^2+H_p^2H_q^2\right)^2}+\frac{(G_pH_qA_m+G_qH_pB_m)^2-H_p^2H_q^2}{\left((G_pH_qA_m+G_qH_pB_m)^2+H_p^2H_q^2\right)^2}-$$

$$\frac{2A_m\left(G_q\left(G_pH_qA_m+G_qH_pB_m\right)+H_pH_q^2\beta_{m,n}\right)}{H_p\left((G_pH_qA_m+G_qH_pB_m)^2+H_p^2H_q^2\right)\left(G_q^2+H_q^2\beta_{m,n}^2\right)}-\frac{2A_m\left(G_q\left(G_qH_pB_m-G_pH_qA_m\right)+H_pH_q^2\beta_{m,n}\right)}{H_p\left((-G_pH_qA_m+G_qH_pB_m)^2+H_p^2H_q^2\right)\left(G_q^2+H_q^2\beta_{m,n}^2\right)}$$